Topic Name:

Method of Moments;

Definition:

The rth moment about origin is the mean of rth power of the deviation of the

Value from zero.

This is for discrete distribution and

$$\mu_r' = \frac{\sum (x)^r}{n}$$

$$\mu_1' = \frac{\sum (x)}{n}$$

$$\mu_2' = \frac{\sum (x)^2}{n}$$

This is for continous probability distribution

$$\mu_1' = E(X) = \int_{-\infty}^{+\infty} x f(x) dx$$

$$\mu_1' = E(X) = \int_{-\infty}^{+\infty} x^2 f(x) dx$$

and these are population moments and sample moments are denoted as;

$$m_r' = E(x)^r$$

$$M_1' = \sum x / n$$

$$M_2' = \sum x^2 / n$$

Procedure:

Calculate the number of moment from the given distribution for the sample (no of moment distribution should be equal to parameter of distribution)

Then calculate population moments and compare them respectively to find the estimates of the parameters.The estimater derived in such a way are known as moments estimate and method is known as methodof moments.

Question :let $x_{1,}, x_{2.} \ldots\ldots x_n$ be a random sample from N(μ, σ^2).Find moments estimate of μ & σ^2

Solution:

Let by definaton of sample momemnt

$M_r' = E(x)^r$

$M_1' = \sum x/n$ (a)

$M_2' = \sum x^2/n$ (b)

Let by defination of pop moment

$\mu_1' = E(X) = \int_{-\infty}^{+\infty} x f(x) dx$ ©

$$\mu_2' = E(X) = \int_{-\infty}^{+\infty} x^2 f(x)dx$$ (d)

$$\mu_1 = \int_{-\infty}^{+\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2(x-\mu/\sigma)^2} dx$$

$$= \int x e^{-1/2(x-\mu/\sigma)^2} dx$$

$$\therefore z = x - \mu/\sigma$$
$$x - \mu = \sigma z$$
$$x = \sigma z + \mu$$
$$dx = \sigma dz$$

Limits remain same

$$= 1/\sigma\sqrt{2\pi} \int_{-\infty}^{+\infty} (\sigma z + \mu) e^{-1/2z^2} \sigma dz$$

$$= \sigma/\sqrt{2\pi} \int_{-\infty}^{+\infty} z e^{-1/2z^2} dz + \mu/\sqrt{2\pi} \int_{-\infty}^{+\infty} e^{-1/2z^2} dz$$ (even function)

$$= 2\mu/\sqrt{2\pi} \int_{0}^{+\infty} e^{-1/2z^2} dz$$

$$\frac{1}{2} z^2 = w$$
$$z^2 = 2w$$
$$z = \sqrt{2w}$$
$$dz = \sqrt{2}\frac{1}{2}(w)^{\frac{1}{2}-1} dw$$

$$= \frac{2\mu}{\sqrt{\pi}} \frac{1}{\sqrt{2}} \int e^{-w} \frac{1}{2} \sqrt{2} w^{\frac{1}{2}-1} dw$$

$$= \frac{\mu}{\sqrt{\pi}} \int e^{-w} w^{\frac{1}{2}-1} dw$$

Compare with Gamma function

$$\frac{\mu}{\sqrt{\pi}} \frac{1}{\overline{)1/2}(1)^{1/2}}$$

$$\alpha = \frac{1}{2}, \beta = 1$$

$E(x) = \mu$

Compare equation (a) & (c)

$\mu = \frac{\sum x}{n}$

$\tilde{\mu} = \bar{x}$

hence x is the required estimator for μ.

N0w

$\mu_2^{/`} = E(x^2)$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^2 e^{-\frac{1}{2}(x-\mu/\pi)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (z\sigma+\mu)^2 e^{\frac{-1}{2}z^2} \sigma dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} z^2 e^{\frac{-1}{2}z^2} dz + \frac{\mu^2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-1}{2}z^2} dz + \frac{2\mu}{\sqrt{2\pi}} \sigma \int_{-\infty}^{+\infty} z e^{\frac{-1}{2}z^2} dz$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} z^2 e^{\frac{-1}{2}z^2} dz + \frac{2\mu}{\sqrt{2}\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{\frac{-1}{2}z^2} dz$$

$$= \frac{2\sigma^2}{\sqrt{2}\sqrt{\pi}} \int_{-\infty}^{+\infty} 2we^{-w} \sqrt{2}\, \frac{1}{2} (w)^{\frac{1}{2}-1} dw + \frac{2\mu^2}{\sqrt{2}\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-w} \sqrt{2}\, \frac{1}{2} w^{\frac{1}{2}-1} dw$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} w^{\frac{1}{2}+1-1} e^{-w} dw + \frac{\mu^2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} w^{\frac{1}{2}-1} e^{-w} dw$$

Compare with Gamma fumction

$$=\frac{2\sigma^2}{\sqrt{\pi}}\Gamma\frac{1}{2}+1.(1)^{\frac{1}{2}+1}+\frac{\mu^2}{\sqrt{\pi}}\Gamma\frac{1}{2}$$

$$=\frac{2\sigma^2}{\sqrt{\pi}}\frac{1}{2}\Gamma\frac{1}{2}+\mu^2$$

$$E(x^2)=\sigma^2+\mu^2 \qquad \text{(d)}$$

$$\sigma^2+\mu^2=\sum x^2\big/n$$

$$\sigma^2=\sum\frac{x^2}{n}-\mu^2$$

$$\sigma^2=\frac{\sum(x-\mu)^2}{n}$$

$$\hat{\sigma}=\sqrt{\frac{\sum(x-\mu)^2}{n}}$$

Hence $\sqrt{\frac{\sum(x-\mu)^2}{n}}$ is required estimator for σ^2.

Q#2

let $x_{1,}x_2..........x_n$ be a random sample from the distribution having the PDF $\frac{e^{-\theta}\theta^x}{x!}$.find moment estimator for θ.

Solution:

Let by definition of sample moment

$$M_{lr}^{l}=E(x)^r$$

$$M_1^l=\sum x/n \qquad \text{(a)}$$

By def of pop moment

$$\mu_r' = E(x)^r$$

$$\mu_1' = E(x)$$

$$\mu_1' = \sum_{x=0}^{\infty} x \frac{e^{-\theta}\theta^x}{x!}$$

$$= \sum_{x=1}^{\infty} x e^{-\theta} \frac{\theta^{x-1}\theta}{x(x-1)!}$$

$$= e^{-\theta}\theta \sum_{x=0}^{\infty} \frac{\theta^{x-1}}{(x-1)!}$$

$$= e^{-\theta}\theta e^{\theta}$$

$$= \theta \qquad \text{(b)}$$

Now we compare a&b

$$\frac{\sum x}{n} = \theta$$

$$\tilde{\theta} = \bar{x}$$

Hence $\bar{x}$ is the required estimator for the parameter θ.

.3: Let x_1 , x_2 , x_3 ,… x_n be a random sample from the –ve exponential distribution with parameter 'θ' having density function $f(x;\theta) = \theta e^{-\theta x}$ .find the moment estimate of parameter 'θ'.

$let\ by\ def\ of\ sample\ moments$

$$M_r' = E(x)^r$$

$$M_1' = \sum x / n$$

$By\ def\ of\ population\ moments$

$$\mu_r' = E(x)^r$$

$$\mu_1' = E(x) = \int_{-\infty}^{+\infty} x f(x) dx$$

$$= \int_{0}^{+\infty} x \theta e^{-\theta x} dx$$

put

$$\theta x = y \qquad x = y / \theta$$

$$dx = \frac{1}{\theta} dy$$

$as\, x \to 0\ then\ y \to 0$

$x \to \infty\ then\ y \to \infty$

$$\mu_1' = \theta \int_0^\infty \left(\frac{y}{\theta}\right) e^{-y} \frac{1}{\theta} dy$$

$$= \frac{1}{\theta} \int_0^\infty y\, e^{-y} dy$$

$Compairing\ with\ gamma\ function$

$$= \frac{1}{\theta} \overline{)2}$$

$$\mu_1' = \frac{1}{\theta}$$

$Compairing\ M_1'\ and\ \mu_1'$

$$\frac{1}{\theta} = \sum x / n$$

$$\tilde{\theta} = 1 / \sum x / n$$

$$\tilde{\theta} = n / \sum x$$

Hence $n / \sum x$ **is the required estimate for parameter '**θ**'.**

Q.4: Let x_1, x_2, x_3 … x_n be a random sample from the biomomial distribution with parameter n & p **having density function. Find the moment estimate of parameter** n & p.

let by def of sample moments

$$M'_r = E(x)^r$$

$$M'_1 = \sum x \big/ n$$

$$M'_2 = \sum x^2 \big/ n$$

By def of population moments

$$\mu'_r = E(x)^r$$

$$\mu'_1 = E(x) = \sum x\, p(x)$$

$$\mu'_2 = E(x^2) = \sum x^2\ p(x)$$

$$\mu'_1 = \sum x\, p(x)$$

$$= \sum_{x=0}^{n} x \left(\frac{n}{x}\right) p^x q^{n-x} \qquad\qquad x : 0,1,2 \ldots n$$

$$= \sum_{x=1}^{n} x \left(\frac{n}{x}\right)\left(\frac{n-1}{x-1}\right) p^{x-1+1} q^{n-x}$$

$$= \sum_{x=1}^{n} n \left(\frac{n-1}{x-1}\right) p^{x-1}\, p\, q^{n-x}$$

$$= np\sum_{x=1}^{n}\binom{n-1}{x-1}p^{x-1}q^{n-x}$$

$$= np\left[\binom{n-1}{1-1}p^0q^{n-1}+\binom{n-1}{2-1}p^1q^{n-2}+\binom{n-1}{3-1}p^2q^{n-3}+\ldots+\binom{n-1}{n-1}p^{n-1}q^{n-n}\right]$$

$$= np\left[q^{n-1}+(n-1)p^1q^{n-2}++\binom{n-1}{2}p^2q^{n-3}+\ldots+p^{n-1}\right]$$

$$= np(q+p)^{n-1} \qquad \therefore q+p=1$$

$$= np(1)^{n-1}$$

$$\mu_1' = np$$

Compare μ_1' *and* M_1'

$$np = \sum x/m$$

$$\tilde{n}\,\tilde{p} = \bar{x} \qquad (1)$$

$$\mu_2' = \sum x^2 p(x)$$

$$= \sum x^2 p(x) = \sum (x(x-1)+x)p(x)$$

$$= \sum_{x=0}^{n} x(x-1)\binom{n}{x}p^xq^{n-x} + \sum_{x=0}^{n} x\binom{n}{x}p^xq^{n-x} \qquad \therefore \sum_{x=0}^{n} x\binom{n}{x}p^xq^{n-x} = np$$

$$= \sum_{x=2}^{n} x(x-1)\left(\frac{n}{x}\right)\left(\frac{n-1}{x-1}\right)\binom{n-2}{x-2}p^{x-2+2}q^{n-x}+np$$

$$= \sum_{x=2}^{n} n(n-1)\binom{n-2}{x-2}p^{x-2}p^2q^{n-x}+np$$

$$= n(n-1)p^2\sum_{x=2}^{n}\binom{n-2}{x-2}p^{x-2}q^{n-x}+np$$

$$= n(n-1)p^2\left[\binom{n-2}{2-2}p^0q^{n-2}+\binom{n-2}{3-2}p^{3-2}q^{n-3}+\binom{n-2}{4-2}p^{4-2}q^{n-4}+\ldots+\binom{n-2}{n-2}p^{n-2}q^{n-n}+np\right]$$

$$= n(n-1)p^2\left[q^{n-2}+(n-2)p^1q^{n-3}++\binom{n-2}{2}p^2q^{n-4}+\ldots+p^{n-2}+np\right]$$

$$= n(n-1)p^2(q+p)^{n-2}+np \qquad \therefore q+p=1$$

$$= n(n-1)p^2+np$$

Compairing μ_2' *and* M_2'

$$\sum x^2/m = \tilde{n}(\tilde{n}-1)\tilde{p}^2+\tilde{n}\tilde{p} \qquad (2)$$

put $\qquad \therefore (eq1)\ \ \tilde{n} = \dfrac{\bar{x}}{\tilde{p}}$

$$\sum x^2 / m = \frac{\bar{x}}{\tilde{p}}\left(\frac{\bar{x}}{\tilde{p}} - 1\right)\tilde{p}^2 + \frac{\bar{x}}{\tilde{p}}\tilde{p}$$

$$\sum x^2 / m = \bar{x}\frac{(\bar{x} - \tilde{p})}{\tilde{p}}\tilde{p}. + \bar{x}$$

$$\sum x^2 / m = \bar{x}(\bar{x} - \tilde{p}) + \bar{x}$$

$$\sum x^2 / m = \bar{x}^2 - \bar{x}\tilde{p} + \bar{x}$$

$$\bar{x}\tilde{p} = \bar{x}^2 + \bar{x} - \sum x^2 / m$$

$$\tilde{p} = \frac{\bar{x}^2 + \bar{x} - \sum x^2 / m}{\bar{x}}$$

$$put \;\; in \;\; eq \;\; (1) \qquad \tilde{n}\tilde{p} = \bar{x}$$

$$\tilde{n}\frac{\bar{x}^2 + \bar{x} - \sum x^2 / m}{\bar{x}} = \bar{x}$$

$$\tilde{n}\left(\bar{x}^2 + \bar{x} - \sum x^2 / m\right) = \bar{x}^2$$

$$\tilde{n} = \bar{x}^2 \Big/ \left(\bar{x}^2 + \bar{x} - \sum x^2 / m\right)$$

Hence $\bar{x}^2 \Big/ \left(\bar{x}^2 + \bar{x} - \sum x^2 / m\right)$ **,** $\frac{\bar{x}^2 + \bar{x} - \sum x^2 / m}{\bar{x}}$ **are the required moment estimate for n & p.**

Q#5

Statement: let $x_{(1)} < x_{(2)} < x_{(3)} < x_{(4)}$ be the order statistic from the distribution having the density function

$$f(x;\alpha) = \frac{1}{\alpha} \qquad\qquad 0 \le x \le \alpha$$

Find the values of a, b, c, d. such that $ax_{(1)}, bx_{(2)}, cx_{(3)} \;\&\; dx_{(4)}$

all are unbiased estimators of α.also find the variance of all estimators.

Solution:

As the ith order statistic is:

$$g(x_i)=\frac{n!}{(i-1)!(n-i)!}[F(x_i)]^{i-1}[1-F(x_i)]^{n-i}f(x_i)$$

Put i=1, n=4

$$g(x_1)=\frac{4!}{(0)!(4-1)!}[F(x_1)]^{0}[1-F(x_1)]^{3}f(x_1)$$

$$=4[1-F(x_1)]^{3}f(x_1)$$

As, the given density function is:

$$f(x)=\frac{1}{\alpha}$$

$$F(x_1)=\int_0^x\frac{1}{\alpha}d(x_1)=\frac{x_1}{\alpha}$$

Now

$$g(x_1)=4\left[1-\frac{x_1}{\alpha}\right]^3\frac{1}{\alpha}$$

$$g(x_{(1)})=\frac{4}{\alpha}\left[1-\frac{x_{(1)}}{\alpha}\right]^3$$

For mean:

$$E(ax_{(1)})=\int ax_{(1)}g(x_{(1)})d(x_{(1)})$$

$$= a\int_0^{\alpha} x_{(1)} \frac{4}{\alpha}\left[1-\frac{x_{(1)}}{\alpha}\right]^3 dx_{(1)}$$

$$= \frac{4\alpha}{\alpha}\int_0^{\alpha} x_{(1)}\left[1-\frac{x_{(1)}}{\alpha}\right]^3 dx_{(1)}$$

Put

$$\frac{x_{(1)}}{\alpha} = w \Rightarrow w\alpha = x_{(1)}$$
$$dx_{(1)} = \alpha dw$$

$$as \quad x_{(1)} \to 0 \ then\ w \to 0$$
$$as \quad x_{(1)} \to \alpha\ then\ w \to 1$$

$$E\left(ax_{(1!)}\right) = \frac{4\alpha}{\alpha}\int_0^1 \alpha w(1-w)^3 \alpha dw$$

$$\alpha = 4\alpha\int_0^1 \alpha w(1-w)^3 dw \qquad\qquad E\left(ax_{(1!)}\right) = \alpha$$

$$\frac{1}{4} = \alpha\int_0^1 w^{2-1}(1-w)^{4-1} dw \qquad (1)$$

As we know the beta function of kind first is:

$$\beta(a,b) = \int_0^1 w^{a-1}(1-w)^{b-1} dw \qquad (2)$$

Comparing (1) & (2)

$a = 2 \quad , b = 4$

$$\frac{1}{4} = a\beta(2,4)$$

$$= a \frac{\Gamma 2 \Gamma 4}{\Gamma 6}$$

$$= a \frac{1!3!}{5!}$$

$$\frac{1}{4} = \frac{a}{20}$$

$5 = a$

$a = 5$

Now we put the value of a & find mean

$$E\left(ax_{(1)}\right) = \int_0^\alpha ax_{(1)} \frac{4}{\alpha}\left[1 - \frac{x_{(1)}}{\alpha}\right]^3 dx_{(1)} \qquad \therefore a = 5$$

$$= \frac{4*5}{\alpha}\int_0^\alpha x_{(1)}\left[1 - \frac{x_{(1)}}{\alpha}\right]^3 dx_{(1)}$$

$$= \frac{20}{\alpha}\int_0^\alpha ax_{(1)}\left[1 - \frac{x_{(1)}}{\alpha}\right]^3 dx_{(1)}$$

$$put\ \frac{x_{(1)}}{\alpha} = \theta \quad , \quad x_{(1)} = \theta d \quad , \quad dx_{(1)} = \theta d\alpha \qquad 0 \le \theta \le 1$$

$$= \frac{20}{\alpha}\int_0^\alpha \theta\alpha[1-\theta]^3 \alpha d\theta$$

$$= 20\alpha\int_0^\alpha \theta^{2-1}[1-\theta]^{4-1} d\theta$$

Comparing with beta function of kind 1st

$$E\left(ax_{(1)}\right) = 20\alpha \ \ \beta(2 \ , \ 4)$$

$$= 20\alpha \ \frac{1! \ 3!}{5!}$$

$$= \frac{20}{20}\alpha$$

$$E\left(ax_{(1)}\right) = \alpha$$

Hence proved

Variance: $E\left(ax_{(1)}\right)^2 - \left[Eax_{(1)}\right]^2$

$$E\left(ax_{(1)}\right)^2 = \int ax_{(1)} g(x_{(1)}) d(x_{(1)})$$

$$= a^2 \int_0^{\alpha} x_{(1)}^2 \frac{4}{\alpha}\left[1 - \frac{x_{(1)}}{\alpha}\right]^3 dx_{(1)}$$

$$= \frac{4*5^2}{\alpha} \int_0^{\alpha} x_{(1)}^2 \left[1 - \frac{x_{(1)}}{\alpha}\right]^3 dx_{(1)}$$

$$= \frac{100}{\alpha} \int_0^{\alpha} x_{(1)}^2 \left[1 - \frac{x_{(1)}}{\alpha}\right]^3 dx_{(1)}$$

Put

$$\frac{x_{(1)}}{\alpha} = w \Rightarrow w\alpha = x_{(1)}$$

$$dx_{(1)} = \alpha dw$$

$$as \quad x_{(1)} \to 0 \ \ then \ w \to 0$$

$$as \quad x_{(1)} \to \alpha \ then \ w \to 1$$

Now

$$E\left(ax_{(1)}\right)^2 = \frac{100}{\alpha} \int_0^1 (w\alpha)^2 [1-w]^3 \alpha dw$$

$$= 100\alpha^2 \int_0^1 (w)^2 [1-w]^3 dw$$

$$= 100\alpha^2 \int_0^1 (w)^{3-1} [1-w]^{4-1} dw$$

Comparing with beta function of kind 1st

$$E\left(ax_{(1)}\right)^2 = 100\alpha^2 \;\; \beta(3,4)$$
$$= 100\alpha^2 \;\; {2!\,3!}\big/{6!}$$
$$= {1200\alpha^2}\big/{720}$$
$$E\left(ax_{(1)}\right) = {120\alpha^2}\big/{72}$$
$$\operatorname{var}\left(ax_{(1)}\right) = {120\alpha^2}\big/{72} - (\alpha)^2$$
$$= \alpha^2\left[\frac{120}{72} - 1\right]$$
$$= \alpha^2\left[\frac{120-72}{72}\right]$$
$$\operatorname{var}\left(ax_{(1)}\right) = {48\alpha^2}\big/{72}$$

b) Now we put i=2, n=4 in ith order statistic:

$$g(x_i) = \frac{n!}{(i-1)!(n-i)!}\left[F(x_i)\right]^{i-1}\left[1-F(x_i)\right]^{n-i} f(x_i)$$
$$g(x_1) = \frac{4!}{(1)!(2)!}\left[F(x_2)\right]^1\left[1-F(x_2)\right]^2 f(x_2)$$
$$= 12\left[F(x_2)\right]\left[1-F(x_2)\right]^2 f(x_2) \qquad (1)$$

As, the given density function is:

$$f(x) = \frac{1}{\alpha}$$
$$F(x_2) = \int_0^x \frac{1}{\alpha} d(x_2) = \frac{x_{(2)}}{\alpha}$$
$$put \;\; in \;\; eq \; (1)$$
$$g(x_2) = 12\left[\frac{x_2}{\alpha}\right]\left[1 - \frac{x_2}{\alpha}\right]^2 \frac{1}{\alpha}$$

For mean:

$$E\left(bx_{(2)}\right)=\int bx_{(2)}g(x_{(2)})d(x_{(2)})$$

$$=b\int_0^{\alpha} x_{(2)}\ \frac{12}{\alpha^2}\ x_{(2)}\left[1-\frac{x_{(2)}}{\alpha}\right]^2 dx_{(2)}$$

$$=\frac{12b}{\alpha^2}\int_0^{\alpha} x_{(2)}^2\left[1-\frac{x_{(2)}}{\alpha}\right]^2 dx_{(2)}$$

$$put \quad \frac{x_{(2)}}{\alpha}=w\Rightarrow w\alpha=x_{(2)}$$

$$dx_{(2)}=\alpha dw$$

$$as \quad x_{(2)}\to 0 \ \ then\ w\to 0$$

$$as \quad x_{(2)}\to \alpha \ then\ w\to 1$$

Now

$$E\left(bx_{(2)}\right)=\frac{12b}{\alpha^2}\int_0^1(\alpha w)^2(1-w)^2\alpha dw$$

$$=\frac{12b}{\alpha}\alpha^2\int_0^1 w^{3-1}(1-w)^{3-1}dw \qquad\qquad E\left(bx_{(2)}\right)=\alpha$$

$$\alpha=12b\alpha\int_0^1 w^{3-1}(1-w)^{3-1}dw \qquad (1)$$

Comparing with beta distribution of kind first

$$\beta(a,b)=\int_0^1 w^{a-1}(1-w)^{b-1}dw \qquad (2)$$

$$a=3\ \ ,\ b=3$$

$$=12b \qquad \beta(3,3)$$

$$=12b\ \frac{\overline{)3}\overline{)3}}{\overline{)6}}$$

$$=12b\ \frac{2!2!}{5!}$$

$$=\frac{12}{30}b$$

$$\frac{5}{2}=b$$

$$b=\frac{5}{2}$$

Now we put the value of b

$$E\left(bx_{(2)}\right)=12\left(\frac{5}{2}\right)\alpha\,{}^{\Gamma 3\,\Gamma 3}\!\big/_{\Gamma 6}$$

$$={}^{30}\!\big/_{30}\,\alpha$$

$$E\left(bx_{(2)}\right)=\alpha$$

Variance: $E\left(bx_{(2)}\right)^2-\left[Ebx_{(2)}\right]^2$

$$E\left(ax_{(2)}\right)^2=\int\left(bx_{(2)}\right)^2 g(x_{(2)})d(x_{(2)})$$

$$=b^2\int_0^\alpha x_{(2)}^2\,\frac{12}{\alpha^2}\,x_{(2)}\left[1-\frac{x_{(2)}}{\alpha}\right]^2 dx_{(2)}$$

$$=\frac{\left({}^{5}\!\big/_{2}\right)^2 12}{\alpha^2}\int_0^\alpha x_{(2)}^3\left[1-\frac{x_{(2)}}{\alpha}\right]^2 dx_{(2)}$$

$$=\frac{75}{\alpha^2}\int_0^\alpha x_{(2)}^3\left[1-\frac{x_{(2)}}{\alpha}\right]^2 dx_{(2)}$$

put

$$\frac{x_{(2)}}{\alpha}=w\Rightarrow w\alpha=x_{(2)}$$

$$dx_{(2)}=\alpha dw$$

$$as\quad x_{(2)}\to 0\ \ then\ \ w\to 0$$

$$as\quad x_{(2)}\to \alpha\ \ then\ \ w\to 1$$

NOW

$$E\left(bx_{(2)}\right)^2=\frac{75}{\alpha}\int_0^1 (w\alpha)^3\left[1-w\right]^2\alpha dw$$

$$=75\alpha^2\int_0^1 (w)^{4-1}\left[1-w\right]^{3-1}dw$$

comparing with beta function of kind first

$$E\left(bx_{(2)}\right)^2 = 75\alpha^2 \ \beta(4,3)$$

$$= 75\alpha^2 \ {3!2!}/{6!}$$

$$= {900\alpha^2}/{720}$$

$$E\left(bx_{(2)}\right) = {5}/{4}\alpha^2$$

$$\operatorname{var}\left(bx_{(2)}\right) = {5\alpha^2}/{4} - (\alpha)^2$$

$$= \alpha^2\left[\frac{5}{4} - 1\right]$$

$$= \alpha^2\left[\frac{5-4}{4}\right]$$

$$\operatorname{var}\left(bx_{(2)}\right) = {\alpha^2}/{4}$$

c) Now we put i=3, n=4 in ith order statistic:

$$g(x_i) = \frac{n!}{(i-1)!(n-i)!}\left[F(x_i)\right]^{i-1}\left[1 - F(x_i)\right]^{n-i} f(x_i)$$

$$g(x_3) = \frac{4!}{(2)!(1)!}\left[F(x_3)\right]^2\left[1 - F(x_3)\right]^2 f(x_3)$$

$$= 6\left[F(x_3)\right]^2\left[1 - F(x_3)\right] f(x_3) \qquad (1)$$

As the given density function is:

$$f(x) = \frac{1}{\alpha}$$

$$F(x_3) = \int_0^x \frac{1}{\alpha} d(x_3) = \frac{x_{(3)}}{\alpha}$$

put in eq (1)

$$g(x_3) = 6\left[\frac{x_3}{\alpha}\right]^2\left[1 - \frac{x_3}{\alpha}\right]\frac{1}{\alpha}$$

$$g(x_3) = \frac{6}{\alpha^3}[x_3]^2\left[1 - \frac{x_3}{\alpha}\right]$$

For mean:

$$E\left(cx_{(3)}\right)=\int cx_{(3)}\,g(x_{(3)})\,d(x_{(3)})$$

$$=\frac{6}{\alpha^3}c\int_0^{\alpha} x_{(3)}x_{(3)}^2\left[1-\frac{x_{(3)}}{\alpha}\right]dx_{(3)}$$

$$=\frac{6}{\alpha^3}c\int_0^{\alpha} x_{(3)}^3\left[1-\frac{x_{(3)}}{\alpha}\right]^2 dx_{(3)}$$

$$put \quad \frac{x_{(3)}}{\alpha}=w \Rightarrow w\alpha = x_{(3)}$$

$$dx_{(3)}=\alpha dw \qquad\qquad 0\le w\le 1$$

$$as \quad x_{(3)}\to 0 \;\; then \;\; w\to 0$$

$$as \quad x_{(3)}\to \alpha \;\; then \;\; w\to 1$$

Now

$$E\left(bx_{(2)}\right)=\frac{6}{\alpha^3}c\int_0^1 (\alpha w)^3(1-w)\alpha dw$$

$$\alpha = 6c\alpha\int_0^1 w^{4-1}(1-w)^{2-1}dw \qquad\qquad E\left(cx_{(3)}\right)=\alpha$$

$$\frac{1}{6c}=\int_0^1 w^{4-1}(1-w)^{2-1}dw \qquad (1)$$

Comparing with beta distribution of first kind

$$\beta(a,b)=\int_0^1 w^{a-1}(1-w)^{b-1}dw \qquad (2)$$

$$a=4 \;\;, b=2$$

$$\frac{1}{6c}=\;\beta(4,2)$$

$$\frac{1}{6c}=\frac{\overline{)4}\,\overline{)2}}{\overline{)6}}$$

$$\frac{1}{6c}=\;3!1!/5!$$

$$=\frac{10}{3}c$$

$$c=\frac{10}{3}$$

Now we put the value of c

$$E\left(cx_{(3)}\right)=6\left(\frac{10}{3}\right)\alpha\frac{\overline{)4}\,\overline{)2}}{\overline{)6}}$$

$$=\frac{20}{20}\alpha$$

$$E\left(cx_{(3)}\right)=\alpha$$

Variance: $E\left(cx_{(3)}\right)^2-\left[Ecx_{(3)}\right]^2$

$$E\left(cx_{(3)}\right)^2=\int\left(cx_{(3)}\right)^2 g(x_{(3)})d(x_{(3)})$$

$$=c^2\int_0^{\alpha}x_{(3)}^2\,\frac{6}{\alpha^3}\,x_{(3)}^2\left[1-\frac{x_{(3)}}{\alpha}\right]dx_{(3)}$$

$$=\frac{\left(10/3\right)^2 6}{\alpha^3}\int_0^{\alpha}x_{(3)}^4\left[1-\frac{x_{(3)}}{\alpha}\right]dx_{(3)}$$

$$=\frac{200}{3\alpha^3}\int_0^{\alpha}x_{(3)}^4\left[1-\frac{x_{(3)}}{\alpha}\right]dx_{(3)}$$

put

$$\frac{x_{(3)}}{\alpha}=w\Rightarrow w\alpha=x_{(3)}$$

$$dx_{(3)}=\alpha dw$$

$$as \quad x_{(3)}\to 0 \;\; then \;\; w\to 0$$

$$as \quad x_{(3)}\to \alpha \;\; then \;\; w\to 1$$

NOW

$$E\left(cx_{(3)}\right)^2=\frac{200}{3\alpha}\int_0^{1}(w\alpha)^4[1-w]\alpha dw$$

$$=\frac{200\alpha^2}{3}\int_0^{1}(w)^{5-1}[1-w]^{2-1}dw$$

comparing with beta function of kind first

$$E\left(cx_{(3)}\right)^2 = \frac{200\alpha^2}{3} \quad \beta(5,\ 2)$$

$$= \frac{200\alpha^2}{3} \quad 4!1!/6!$$

$$= 200\alpha^2/90$$

$$E\left(cx_{(3)}\right) = 200/90\,\alpha^2$$

$$\operatorname{var}\left(cx_{(3)}\right) = 5\alpha^2/9 - (\alpha)^2$$

$$= \alpha^2\left[\frac{200}{90} - 1\right]$$

$$= \alpha^2\left[\frac{200-90}{90}\right]$$

$$= \alpha^2\,\frac{110}{90}$$

$$\operatorname{var}\left(cx_{(3)}\right) = \frac{11\alpha^2}{9}$$

d) Now we put i=4, n=4 in ith order statistic:

$$g(x_i) = \frac{n!}{(i-1)!(n-i)!}\left[F(x_i)\right]^{i-1}\left[1-F(x_i)\right]^{n-i} f(x_i)$$

$$g(x_4) = \frac{4!}{(3)!(0)!}\left[F(x_4)\right]^3\left[1-F(x_4)\right]^0 f(x_4)$$

$$= 4\left[F(x_4)\right]^3 f(x_4) \qquad (1)$$

As the given density function is:

$$f(x) = \frac{1}{\alpha} \quad \Rightarrow \quad f(x_{(4)}) = \frac{1}{\alpha}$$

$$F\left(x_{(4)}\right) = \int_0^x \frac{1}{\alpha}\,d(x_4) = \frac{x_{(4)}}{\alpha}$$

$$put \;\; in \;\; eq \; (1)$$

$$g\left(x_{(4)}\right) = 4\left[\frac{x_4}{\alpha}\right]^3 \frac{1}{\alpha}$$

$$g\left(x_{(4)}\right) = \frac{4}{\alpha^4}\left[x_4\right]^3$$

For mean:

$$E\left(dx_{(4)}\right)=\int dx_{(4)}g(x_{(4)})d(x_{(4)})$$

$$=d\int_0^{\alpha} x_{(4)}\frac{4}{\alpha^4}\left(x_{(4)}^3\right)dx_{(4)}$$

$$=\frac{4}{\alpha^4}d\int_0^{\alpha} x_{(4)}^4 dx_{(4)}$$

$$put \quad \frac{x_{(4)}}{\alpha}=w \Rightarrow w\alpha = x_{(4)}$$

$$dx_{(4)}=\alpha dw \qquad\qquad 0\le w\le 1$$

$$as \quad x_{(4)}\to 0 \;\; then \; w\to 0$$

$$as \quad x_{(4)}\to \alpha \;\; then \; w\to 1$$

Now

$$E\left(dx_{(4)}\right)=\frac{4}{\alpha^4}d\int_0^1 (\alpha w)^4\alpha dw$$

$$\alpha=\frac{4d}{\alpha^3}\alpha^4\int_0^1 w^{4-1}dw \qquad\qquad E\left(dx_{(4)}\right)=\alpha$$

Variance: $E\left(dx_{(4)}\right)^2-\left[Edx_{(4)}\right]^2$

$$E\left(dx_{(4)}\right)^2=\int\left(dx_{(4)}\right)^2 g(x_{(4)})d(x_{(4)})$$

$$=d^2\int_0^{\alpha} x_{(4)}^2\frac{4}{\alpha^4}x_{(4)}^3 dx_{(4)}$$

$$=\frac{\left(5/4\right)^2 4}{\alpha^4}\int_0^{\alpha} x_{(4)}^5 dx_{(4)}$$

$$=\frac{25}{4\alpha^4}\frac{x_{(4)}^6}{6}\Big|_0^{\alpha}$$

$$=\frac{25}{4\alpha^4}\frac{\alpha^6}{6}$$

$$E\left(dx_{(4)}\right)^2=\frac{25}{24}\alpha^2$$

$$\operatorname{var}\left(dx_{(4)}\right) = \frac{25}{24}\alpha^2 - \alpha^2$$

$$= \alpha^2\left[\frac{25}{24} - 1\right]$$

$$= \alpha^2\left[\frac{25-24}{24}\right]$$

$$\operatorname{var}\left(dx_{(4)}\right) = \frac{\alpha^2}{24}$$

Q#6

Statement:

Show that a sample from

$$f(x) = \frac{1}{\overline{)\rho}\theta^{\rho}} x^{\rho-1} e^{-x/\theta}$$

Where

$\rho > 0\, and \quad 0 < x < \infty$, Both are inclusive. Then find uniform variance unbiased estimator of θ is $\frac{\bar{x}}{\rho}$ with variance $\theta^2/n\rho$. Also find the values using the given data.

X:
22,23,22.5,23,24,22,25,23.5,22,23,22,24,23,25,22,22,26, 23,24,22,22.5,23,23,22,24,22,25,23,23,22.

b) Show that $\Sigma \log x_i / n$ is minimum variance bound unbiased estimator of $\partial \log L(x) / \partial\rho$ with variance $\left(\frac{\partial^2 \log L(x)}{\partial\rho^2}\right)^2 \Big/ n$ with $\theta = 1$ (assume).Also find the value of MVBUE by using the above values when $\rho = 0.75$.

Solution:

As,

$$f(x)=\frac{1}{\overline{)\rho}\theta^{\rho}}x^{\rho-1}e^{-x/\theta}$$

Applying likelihood

$$L(x)=\left(\frac{1}{\overline{)\rho}\theta^{\rho}}\right)^{n}\left(\prod_{i=1}^{n}x\right)^{\rho-1}e^{-\sum x/\theta}$$

Taking log on both sides

$$\log L(\underline{x})=n\log\left(\frac{1}{\overline{)\rho}\theta^{\rho}}\right)+(\rho-1)\sum\log x-\frac{\sum x}{\theta}$$

$$=-n\log.\overline{)\rho}-n\log\theta^{\rho}+(\rho-1)\sum\log x-\frac{\sum x}{\theta}$$

$$=-n\log\overline{)\rho}-n\rho\log\theta+(\rho-1)\sum\log x-\frac{\sum x}{\theta}\qquad (A)$$

Differentiate w.r.t to θ:

$$\partial\log L(x)/\partial\rho=0-\frac{n\rho}{\theta}+\frac{\sum x}{\theta^{2}}$$

$$=\frac{\sum x}{\theta^{2}}-\frac{n\rho}{\theta}$$

$$=\frac{n\rho}{\theta^{2}}\left[\frac{\sum x}{n\rho}-\theta\right]$$

$$=\frac{n\rho}{\theta^{2}}\left[\frac{\bar{x}}{\rho}-\theta\right]$$

Comparing with general equation:

$$A(\theta)=\left[\hat{\theta}-\tau(\theta)\right]$$

$$A(\theta)=\frac{n\rho}{\theta^{2}}\quad,\quad\hat{\theta}=\frac{\bar{x}}{\rho}\quad,\quad\tau(\theta)=\theta\quad,\quad\tau'(\theta)=1$$

$$\text{var}(\theta) = \tau'(\theta) \Big/ A(\theta)$$

$$= 1 \Big/ \frac{n\rho}{\theta^2}$$

$$\text{var}(\theta) = \theta^2 \big/ n\rho$$

Hence $\bar{x}$ is MVBUE of θ with variance $\frac{\theta^2}{n\rho}$

X	**Log X**
22	1.3424
23	1.3617
22.5	1.3522
23	1.3617
24	1.3802
22	1.3424
25	1.3979
23.5	1.3711
22	1.3424
23	1.3617
22	1.3424
24	1.3802
23	1.3617
25	1.3979
22	1.3424
22	1.3424
26	1.4150
23	1.3617
24	1.3802
22	1.3424
22.5	1.3522
23	1.3617
23	1.3617
22	1.3424
24	1.3802
22	1.3424
25	1.3979
23	1.3617
23	1.3617
22	1.3424
692.5	**40.8843**

$p = 0.75$, $\sum x = 692.5$, $\sum \log x = 40.8843$, $n = 3$

$$\bar{x} = 23.08$$

$$\hat{\theta} = \frac{\bar{x}}{\rho} = 30.77$$

$$\operatorname{var}(\hat{\theta}) = \frac{\hat{\theta}^2}{n\rho} = 23.66$$

Differentiate eq a w.r.t. ρ

$$\frac{\partial \log L(\underline{x})}{\partial \rho} = \frac{-n\partial \log \overline{)\rho}}{\partial \rho} - n\log\theta - \sum \log x - 0$$

put $\theta = 1$

$$= \frac{-n\partial \log \overline{)\rho}}{\partial \rho} + \sum \log x$$

$$= \sum \log x - \frac{n\partial \log \overline{)\rho}}{\partial \rho}$$

$$= n\left[\frac{\sum \log x}{n} - \frac{n\partial \log \overline{)\rho}}{\partial \rho}\right] \rightarrow 1$$

as we know the general eq is:

$$= A(\theta)\left[\hat{\theta} - \tau(\theta)\right] \rightarrow 2$$

compering eq 1 *and* 2:

$$A(\theta) = n \ , \hat{\theta} = \frac{\sum \log x}{n} \quad , \ \tau(\theta) = \frac{\partial \log \overline{)\rho}}{\partial \rho} \qquad \tau'(\theta) = \frac{\partial^2 \log \overline{)\rho}}{\partial \rho^2}$$

Hence $\dfrac{\sum \log x}{n}$ *is* min*imum bound unbiased estimator of* θ.

$$\hat{\theta} = \frac{\sum \log x}{n} = 1.36$$